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Methodological Imperfection and Formalizations of Scientific Activity

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Any mathematical formalization of scientific activity allows for imperfections in the methodology that is formalized. These can be of three types, "dirty," "rotten," and "dammed." Restricting mathematical attention to those methods that cannot be construed to be imperfect drastically reduces the class of objects that must be analyzed, and relates all other objects to these more regular ones. Examples are drawn from empirical logic.

1. INTRODUCTION

Nothing is perfect. Few contest this, yet most reach for perfection by trying harder. In scientific investigations this situation manifests itself on the one hand by a never-ending effort to achieve more precise and clear-cut observations and experiments, and on the other hand by using ever more developed analyses of errors. On top of this, the theorists, hoping for abstract perfection, introduce idealizations, achieving in this way, they believe, abstract objects more manageable than would be the case if one kept closer to actual scientific activity. The past half century has seen a development, under various schools of thought, of a mathematical metatheory of science: formalizations and idealizations of scientific activity itself. This has been greatly motivated by a desire to come to terms with quantum mechanics. By a formalization of science we mean any mathematical scheme that describes acts of observation, and by specification, the more restricted acts of experimentation. The whole of statistics and a respectable portion of quantum mechanics exemplify such concerns. The recent more systematic formalizations try to give these concerns a solid mathematical and philosophical footing. The intent of course is somehow to capture the essence of legitimate and effective scientific procedures with as many *a priori* assumptions laid bare as possible.

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In this effort, though, two things get overlooked. Once one has a general enough formalism, then along with scientific activity, the formalism admits as examples activities that are far from scientific. Divinatory procedures from cartomancy to astrology have strong formal resemblance to scientific activity. Even arbitrary acts that mimic science without any relation to a field of phenomena are subsumed in such formalisms. One thing that distinguishes these activities from truly scientific ones is the second fact that is overlooked in the formalizations: the constant effort of any science to disprove itself. It is an essential part of any science to try to find alternative explanations for each experiment and observation. If some experimenter is lax in this, then his or her colleagues certainly will not be. Only when no alternative within the currently accepted schemes is at hand is the result accepted for the moment. This attitude and activity is not usually included in the formalism. What is essential to include such concerns is the notion of imperfection. Imperfection in scientific methodology takes many forms: our apparatus and methods are imperfect, our data analysis is imperfect, the state of affairs we study is imperfectly prepared or identified, and our knowledge, both empirical and theoretical, is imperfect. Even more seriously, our approach to any phenomenal field may be inappropriate to it, yielding irrelevant and useless data, or, at best, data highly contaminated by our prejudices and ignorance. Given now a formalization of science, these facts must somehow be expressed and analyzed and conclusions drawn from them. In this paper we point out ways that this can be done, drawing mostly for examples on what may be grossly labeled "quantum logic." We have only a modicum of hard results. The purpose of this paper is more to call attention to these problems and to indicate what one may accomplish by their study. The actual mathematical analysis proposes a set of difficult questions to which only very partial answers are now known. We keep the exposition informal, placing all the mathematical technicalities in an Appendix. A familiarity with the author's works (Svetlichny, 1981, 1982), is useful, but not essential.

2. IMPERFECTION AS ADULTERATION OF DATA

As scientists we believe there *is* something to be scientific about, that we *can* know something about it, and that we *do* have effective methods for obtaining this knowledge. Thus, even to begin we must make a metaphysical assumption that these three aspects make sense: object, knowledge, and method. In relation to the method, we distinguish what we can call the laboratory attitude from the field attitude. In the laboratory attitude one assumes that the methods are sufficiently refined that one can contemplate a series of well-defined acts that lead to one out of a previously defined set of outcomes. In making a voltage measurement in the laboratory one takes the voltmeter reading as the results of the experiment, and ignores the cat fight outside as being just part of the irrelevant background. The field attitude is less restrictive, having no *a priori* set of possible results from which to choose; one simply notes down what seems significant, deciding on what is relevant by more lax criteria. This is an indispensable approach to science, but it is not yet well formalized, and here we shall only treat the laboratory attitude.

The general thesis we want to make explicit here is the following: once one has a specific formalization of the above three aspects, making it a model of scientific activity, then one has committed oneself to the existence of *some* specific type of primitive object. Once these objects must be admitted as existing in the world, they can partake in the methodology, become imbedded so to speak in the method, and take part in the production of data. The data will then not only reflect information about some object of study, but also incorporate adulterating contributions from the imbedded primitive objects. This leads to imperfections in the method, which we must try to minimize. Of course, one cannot by purely formal criteria *prove* that a method *is* flawed in such a manner, but it is quite fortunate that one can establish criteria for when a method *may be* so flawed, and by exclusion, criteria for when the method is *certainly not* so flawed. This then separates from a plethora of possible mathematical objects a small subset of those that may be called flawless, and which we believe must constitute the primary focus of mathematical investigation.

We shall show that there are three ways that a method may be thought to be imperfect. These could be vulgarly labeled as "dirty," "rotten," and "damned." The first type obtains whenever a method can be interpreted as the result of using a different method and then subjecting the results to an information-adulterating processing. The second type obtains whenever a method can be interpreted as composed of component methods in an information-adulterating way. The third type obtains whenever the given method is flawless in the first two senses, but the formalism allows for conceivable but nonexistent methods in terms of which the given method can now be construed as flawed. It is as though Nature were cheating us of a better view, restricting our abilities in a way not directly explicable by the formalism.

3. PROBABILISTIC EMPIRICAL AND QUANTUM LOGICS

In the laboratory attitude, each method when executed yields one out of a predefined set of *outcomes*. Identify such a method E with its set of outcomes: $=\{a, b, \ldots\}$. A collection of methods is thus given by a set of sets $\mathfrak{A}$ and is called a *quasimanual*. Each $E \in \mathfrak{A}$ is called an *operation*. A subset $A \subset E$ of an operation is called an *event*. *Empirical logic* is the study of quasimanuals, their properties, and interpretations (Foulis and Randall, 1978, 1981; Randall and Foulis, 1978). The probabilistic interpretation of quasimanuals assumes that given an ontological state of affairs and an operation E, the execution of E leads to some outcome $x \in E$ with a certain probability $w(x)$, usually considered as an asymptotic frequency. When x belongs to several operations, this probability is assumed to be independent of the operation that is executed. In a probabilistic view of scientific prediction, this equality in all ontological states is usually taken to be a necessary (though not alway sufficient) condition for outcome identification of different operations. One thus defines a map w from the outcome set to $[0, 1]$ such that for all $E \in \mathfrak{A}$ one has $\sum_{x \in E} w(x) = 1$. Such maps are called *weights*.

A *quantum logic*, which is an orthomodular poset L, falls into this scheme. Let $\mathfrak{A}$ be the set of *finite orthopartitions of unity*; that is, finite subset of mutually orthogonal elements whose supremum is 1. Weights are now finitely additive probability measures on L. One knows that L can be recovered from $\mathfrak{A}$ in this case.

One must note now that having assumed a probabilistic interpretation, for it to be useful one must admit the existence of stochastic processes in the world. One must also admit that the notion of a series of independent experiments makes sense. Even if the phenomena one studies are not to be considered independent, for any sampling to be treatable, the statistician must have at hand a process whose repetition can be considered as independent. One must thus posit as existing in the world a primitive object acting as a stochastic process, the repeated executions of which are to be considered as independent. Given one such, then by mathematical transformations of the outcomes of runs of executions of the process, any other one can be modeled to any degree of accuracy. There is therefore no loss of generality to assume that we have access to finite random variables with values in any given finite set and with any given corresponding probability measure. We assume that independent invocations of these processes are possible, and that the result of each invocation is independent of anything else of relevance in the given circumstances. Such processes we call *stochastic splitters* and we can construe them as operations that can be added to any quasimanual. Given that the existence of stochastic splitters is a necessary admission of any probabilistic view of scientific methodology, these splitters, according to our thesis, can become imbedded in the methods to adulterate data. The existence of splitters has further consequences. Consider one with outcomes $\{1, 2\}$ and outcome probabilities λ and $1 - \lambda$. Suppose the ontological states of affairs one studies can in fact be prepared at will, and consider two such

S_1 and S_2 giving rise to weights w_1 and w_2. A new state preparation procedure is now given by the following prescription: invoke the splitter; if the result is i, prepare the state S_i. This new procedure leads to weight $\lambda w_1 + (1-\lambda) w_2$, and so the set W of weights corresponding to preparation procedures can be taken to be convex. We shall always assume this in the sequel.

4. FREQUENCY FUNCTIONALS AND INSTRUMENTAL THEORIES

Let W be a convex set. An affine functional f: $W \to [0, 1]$ we call a *frequency functional.* If one considers that W somehow embodies a set of ontological states of affairs, and if E is some laboratory operation with outcomes $a, b, \ldots$, then the frequency of occurrence of each of these outcomes defines a frequency functional on W and one has a family $(f_x)_{x \in E}$ of frequency functionals. Similarly, if A is an event, then $f_A = \sum_{x \in A} f_x$ is a frequency functional corresponding to the event. Such a narrowing of attention to frequency functionals discards a lot of information, yet the simple arrangement of a convex set with a set of sets of frequency functionals lends itself to many significant mathematical results. The fundamental questions, however, even in such a simplified scheme, pose interesting challenges.

Formally one thus considers systems $T = (W, (O_n)_{n=2,3,\ldots})$, where W is a convex set, which we call the set of *states*, and for each $n \geq 2$, O_n is a convex set of n-tuples of frequency functionals whose components sum to 1. Each $w \in W$ is to be interpreted as corresponding to a preparation procedure of some ontological state of affairs, and each $(f_1, \ldots, f_n) \in O_n$, called an *instrument,* as the family of frequency functionals corresponding to some operation with its outcomes ordered in some given manner.

One can justify the convexity of O_n by being able to condition the choice of an operation on the outcome of a stochastic splitter. A stochastic splitter with outcome probabilities $\lambda_1, \ldots, \lambda_n$ has a representation in O_n as $(\lambda_1 1, \ldots, \lambda_n 1)$, where 1 is the constant frequency functional. Elements of O_2 are of the form $(f, 1-f)$ and the set of f appearing in such pairs is a convex set, which we shall designate by O. One has $O_n \subset O^n$. We shall always assume in the sequel that O separates points of W. We further assume that if $(f_1, \ldots, f_n) \in O_n$ and if one partitions $\{1, \ldots, n\}$ into m subsets $Q_1, \ldots, Q_m$ (some of which may be empty) and set $g_i = \sum \{f_j | j \in Q_i\}$, then $(g_1, \ldots, g_m) \in O_m$. We call this process *coarsening* and also say that $(f_1, \ldots, f_n)$ *refines* $(g_1, \ldots, g_m)$. That $(g_1, \ldots, g_m)$ can be construed as corresponding to an operation can be seen by the following prescription. Let $E = \{e_1, \ldots, e_n\}$ be an operation with frequency functionals $(f_1, \ldots, f_n)$.

If upon executing E the result is e_i with $i \in Q_j$, consider the number j as the outcome of a new operation. This new operation obviously has frequency functionals $(g_1, \ldots, g_m)$. The structure here is what in Svetlichy (1981) we call an *instrumental theory.* Such a structure can be attached to any quasimanual $\mathfrak{A}$: Let W be any convex subset of $\Omega(\mathfrak{A})$, the set of all weights on $\mathfrak{A}$. Let O_n be the convex hull of n-tuples of the form $(f_{A1}, \ldots, f_{An})$, where $(A_1, \ldots, A_n)$ are n-element partitions into not necessarily nonempty events of operations E of $\mathfrak{A}$. Such a theory in which one chooses $W = \Omega(\mathfrak{A})$ we call the *canonical instrumental theory* associated to $\mathfrak{A}$.

Mielnik (1968, 1974) introduced a formalism that has as the starting point a convex set W to be considered as representing the set of ontological states under a probabilistic interpretation. Mielnik assumes that any frequency functional corresponds to some yes-no observation procedure. An even stronger claim would be to consider that each n-tuple of frequency functionals $(f_1, \ldots, f_n)$ whose sum is 1 corresponds to an executable operation. This may sound extravagant, since why should Nature allow access to any such n-tuple? Yet, as we shall see, these theories are central to the study of general ones. A theory $(W, (O_n)_{n=2,3\ldots}))$ constructed in this manner we shall call the *canonical theory associated to W.*

5. THE DIRTY AND THE ROTTEN

Let $T = (W, (O_n)_{n=2,3,\ldots}))$ be an instrumental theory. We say an instrument $J = (g_1, \ldots, g_m)$ is *stochastically factorizable* (Svetlichny, 1982) if there is an instrument $I = (f_1, \ldots, f_n)$ and a nontrivial stochastic matrix $P(j, i)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ $[0 \le P(j, i) \le 1;\ \sum_j P(j, i) = 1;\ 0 < P(a, b) < 1$ for some $a, b]$, such that $g_j = \sum_i P(i, j) f_i$. By Svetlichny (1982) such constructs are allowed once one admits the existence of stochastic splitters. One can now view J as being I followed by an information-adulterating stochastic process. This is an example of a "dirty" instrument. One has from Lemma 1 of Svetlichny (1982) that J is stochastically factorizable if and only if there is an $h \in O$ and a pair of elements, (say g_1, g_2 after renumbering) of J such that

$$(g_1 - h/2, h, g_2 - h/2, g_3, \ldots, g_m) \in O_{m+1}$$

Consider now the canonical theory associated to some convex set W. In this case $(f, 1-f) \in O_2$ is stochastically factorizable if and only if there is an $h \in O$ such that $(f - h/2, h, 1 - f - h/2) \in O_3$. This, however, holds if and only if $f - h/2$ and $1 - f - h/2$ are both in O, which is true if and only if both $f \pm h/2$ are in O. Thus, $(f, 1-f)$ is stochastically factorizable if and only if f can be displaced in either direction by an element of O while still remaining in O. For the case in which W is a square $[0, 1]^2$, O has the geometric form given by Figure 1, where we label each extreme point by the corresponding frequency functional.

Here, for $(x, y) \in [0, 1]^2$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = 1 - x$, and $f_4(x, y) = 1 - y$. One now sees from the above criterion that the set of f such that $(f, 1-f)$ is "clean," that is, not stochastically factorizable, consists of the boundary of the square spanned by the four frequency functionals f_i. Several comments can be made concerning this.

In the first place, convex polytopes for W correspond to situations that are essentially finite, since only a finite number of linear constraints are necessary to define W. One should then expect from any criterion for flawlessness to obtain only a finite number of objects. Thus, lack of stochastic factorizability cannot be considered as the correct notion of perfection. One notes that among the clean instruments one has all the extreme points. Svetlichy (1982) argues that it is the extreme ones that must be considered as not containing information-adulterating stochastic elements. In fact if the instument I is not extreme, then one can write it as $I = \frac{1}{2}I_1 + \frac{1}{2}I_2$ and its operation can be construed as first invoking a splitter and on the basis of the result operating either with I_1 or I_2. Thus, I can be viewed as compound with an embedded stochastic element. This added element introduces extra complexity in successive observed outcomes as compared to the mean of the complexities corresponding to the two component instruments (Svetlichny 1982). Nonextreme instruments thus correspond to the type of imperfection that we call "rotten."

For the canonical theories of convex polytopes one has only a finite number of extreme points in O_n, and this is in accord with our intuition. What is even more in accord with our intuition is that in this case there is a number n_0 such that for $n > n_0$ each extreme point of O_n has at least $n - n_0$ zero functionals (Theorem 1, Appendix). Thus, there is essentially only a *finite* number of extreme instruments. One has from Svetlichny (1982) Lemma 6, that no two nonzero functionals in an extreme instrument are

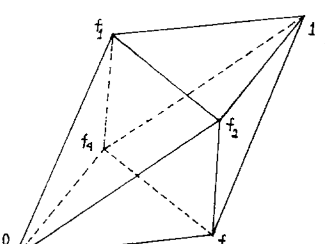

Fig. 1. Frequency functionals on the square.

equal, and so by considering the *sets* $\{f_1, \ldots, f_n\}$ of nonzero frequency functionals corresponding to extreme instruments $(f_1, \ldots, f_n)$, and ignoring for convenience the singleton $\{1\}$, one has a quasimanual $\mathfrak{A}^{ex}(T)$, which we call the *extreme instrument quasimanual* of T.

Having this new quasimanual, one can consider the new set of weights $W' = \Omega(\mathfrak{A}^{ex}(T))$. One has a natural inclusion $W \to W'$, yet W' could be bigger. Whenever $W' = W$ one says the theory is *state complete.* Elements of $W' \setminus W$ can be interpreted as conceivable states allowed by the theory but which for some reason do not exist in actuality or are not accessible to the experimenter, or are for some *ad hoc* reason being ignored. Each f_i of an extreme instrument $I = (f_1, \ldots, f_n)$ can be interpreted as a frequency functional f_i' on W', so the convex span of the $I' = (f_1', \ldots, f_n')$ will form a new instrument set O_n'. The system $T' = (W', (O_n')_{n=2,3,\ldots}))$ forms a new theory, which we call the *state completion* of T. One has $T'' = T'$ (Theorem 2, Appendix). State-incomplete theories can be thought of as being carved out of state-complete ones by restriction on state production (Svetlichny, 1981, Definition 32). In the process, extreme instruments may by statistical coincidence become no longer extreme.

To illustrate these points, consider the case where $W = [0, 1]$ and in which the O_n are to be defined as the smallest sets consistent with the existence of (f_1, f_2, f_3) as an extreme instrument, where $f_1(x) = 1/2, f_2(x) = x/2$, and $f_3(x) = 1/2 - x/2$. One has that $(f_1, f_2 + f_3) = (1/2, 1/2)$ is a non-extreme coarsening of an extreme instrument. This suggests that there is some sort of imperfection in the original instrument. One finds in fact that $\mathfrak{A}^{ex}$ is a trichotomy with two of its coarsenings, and thus W' is a triangle with the original set W imbedded as shown in Figure 2.

The f_i can now be considered as the restrictions of the three barycentric coordinates of the triangle, In relation to the triangle, of course, all the coarsenings of the instrument given by the three barycentric coordinates are extreme. One may hope therefore that by passing to state completions one would achieve a situation in which each extreme operation can be considered as flawless. This, however, is not the case.

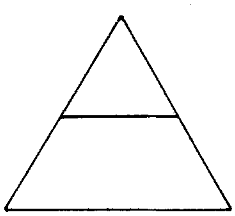

Fig. 2.

6. IMPERFECT COARSENINGS AND THE DAMNED

An extreme instrument with a nonextreme coarsening is *prima facie* a curious situation. Let $I=(f_1,\ldots,f_n)$ be an instrument representing some operation $E=\{e_1,\ldots,e_n\}$ and P_i the proposition that affirms that the ith outcome e_i is realized. Introduce now a partition $A_1,\ldots,A_n$ of $\{1,\ldots,n\}$ and the operation $D=\{d_1,\ldots,d_n\}$, where d_j occurs if and only if, upon executing E, event A_j occurs. Now D has frequency functionals $(f_{A1},\ldots,f_{An})$, which constitute an instrument J that is a coarsening of I. The proposition Q_j that affirms that d_j occurs is equivalent to $\bigvee_{i\in Aj} P_i$, which is a purely logical construct. How can purely logical constructs on outcomes of an experiment that cannot be interpreted as possessing imbedded stochastic elements lead to one that can be so interpreted? The answer that our last example proposed is that the theory may be state incomplete, and so just by pure statistical coincidence in a reduced situation, an extreme instrument is identified with one that is not. However, our situation can occur even in a state-complete theory. To see this, consider the quasimanual

$$\mathfrak{A}=\{\{a,b,x\},\{c,d,x\},\{a,b,c,d\}\}$$

which we call the "fly" and which can be diagrammed as in Figure 3.

One readily shows that for any weight w one has $w(x)=1/2$, so that $w(a)+w(b)=1/2=w(c)+w(d)$ and thus $\Omega(\mathfrak{A})$ is essentially the square $[0,1/2]^2$ defined, say, by the pairs $(w(a),w(c))$. The set O is spanned by the frequency functionals of events, in which one easily finds that f_a, f_b, f_c, and f_d are extreme points. Since each initial instrument has at most one nonextreme outcome, it is extreme. Each O_n is the span of elements of the form $(f_{A1},\ldots,f_{An})$, where $A_1,\ldots,A_n$ is a partition of some operation. Thus, the extreme instruments must be among these elements. One then readily finds that starting from initial instruments, the other extreme ones can be reached by successive coarsenings involving at each step only two outcomes. Thus $w\in W'$ is uniquely determined by its restriction to the initial

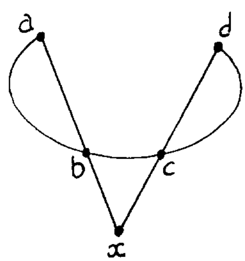

Fig. 3. The fly.

instruments and so the theory is state complete. One has, however, that $(f_a+f_b, f_x)=(1/2, 1/2)$ is a nonextreme coarsening of an extreme instrument. In this case an explanation of the phenomenon can be seen in noting that since the state space is the square, one can imbed the O_n into the corresponding sets of the canonical theory of the square. One then finds that the initial instruments of the fly, though extreme in the canonical theory of the fly, are not extreme in the canonical theory of the square.

The initial instruments are thus flawed, but in relation to presumably unattainable procedures, admitted, however, by the mathematical description. This is an example of a type of imperfection that we have labeled the "damned." One can appreciate this more concretely by the following construction: Take $W=[0, 1]^2$ and define f, g by $f(u, v)=u$ and $g(u, v)=v$. Thus, $(f, 1-f)$ and $(g, 1-g)$ are two extreme dichotomic instruments. Consider now the compound operation diagrammed in Figure 4. This involves a stochastic splitter Z with equal probabilities followed by either the first or the second dichotomy. Identify this with the instrument (f_a, f_b, f_c, f_d) of the fly and the instruments (f_a, f_b, f_x) and (f_c, f_d, f_x) with the obvious coarsenings. We have thus reproduced the instrumental theory of the fly as a compound theory. One must now envisage that the instrument (f_a, f_b, f_c, f_d) is accessible to the experimenter, but Nature somehow conspires to make it impossible to construct $(f, 1-f)$ and $(g, 1-g)$ as self-subsisting procedures in their own right. These can only exist bound within the construct. One can readily understand now why a coarsening of an extreme instrument can be nonextreme, for (f_a+f_b, f_c+f_d) does nothing else but expose the hidden stochastic splitter in the compound instrument. We thus conjecture that whenever a state-complete theory has nonextreme coarsenings of extreme instruments, it incorporates hidden stochastic splitters, that is, some extreme instrument is nonextreme in the corresponding canonical theory associated to the state space. A precise mathematical problem based on this idea is the following:

Conjecture. Let W be a compact, finite-dimensional, convex set, and let $\mathfrak{A}^{\mathrm{ex}}$ be the extreme instrument quasimanual of the corresponding canoni-

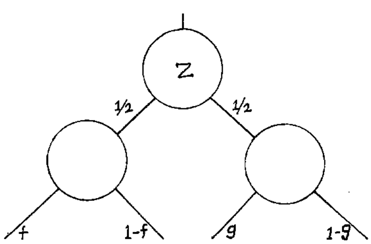

Fig. 4.

cal theory. If $\Omega(\mathfrak{A}^{\mathrm{ex}}) = W$, then for each event A of $\mathfrak{A}^{\mathrm{ex}}$, the frequency functional f_A is an extreme point of O.

Unfortunately, we have been unable to settle this conjecture even in the case of convex polytopes. If false, it would mean that a state-complete canonical theory of some convex set would have an extreme instrument with a nonextreme coarsening. This would be inexplicable in terms of the instruments being nonextreme in a more encompassing theory, since such canonical theories are in a natural sense maximal. This would signal the existence of a more insidious imperfection than the one we have labeled as "damned," and we conjecture that it does not exist.

7. THE IMPORTANCE OF CANONICAL THEORIES

Suppose we have some phenomena that are well described by an instrumental theory $T = (W, (O_n)_{n=2,3,\ldots}))$. Imagine now an experimenter S hired to supply data concerning the phenomena. Due to personal tastes, S refuses to prepare and observe certain states, thus effectively looking only at a subset $W_S \subset W$. Likewise, S refuses to work with certain instruments, and so effectively only works with subsets $O_n^S \subset O_n$ of the instrument sets. Thus, in presenting the results for analysis, one has effectively only the fragment $T_S = (W_S, (O_n^S)_{n=2,3,\ldots})$ instead of T. The various schools of formalization of scientific methodology all claim to have tools to analyze and describe T_S, *but why should one want to*?

Now, T_S lies within T and if not too different (for example, if W_S and W have the same dimension) cannot be a state-complete canonical theory. State-complete canonical theories cannot be interpreted as being achieved just by an *ad hoc* reduction by some unscrupulous experimenter. All other theories are contained within these, and though one cannot formally prove that any such *does* result from an *ad hoc* reduction, it *could* so result. This is sufficient reason to relegate such theories to secondary mathematical analysis and to concentrate on the state-complete canonical ones. A classification of these is thus of greatest importance. This is a very difficult problem, for which only the following result is known: the canonical theory associated to a finite Cartesian product of simplexes is state complete (Theorem 3, Appendix).

8. THE QUALITATIVE INTERPRETATION OF QUASIMANUALS

As further illustrations of our ideas, we apply them to the qualitative interpretation of quasimanuals due to Foulis *et al.* (1983). Given a quasimanual and an ontological state S, let us list for each operation E the results that are possible in that state. The set of all such outcomes taken for all operations thus defines a subset P of the outcome set. We now make a minimal realist assumption. We say that an event A of an operation E is *S-certain in E* if $P \cap E \subset A$. Assume now that the S-certainty of an event is independent of the operation in which the event may lie. Suppose now that for two operations E and F one has $P \cap E \subset F$; then $P \cap E$ is S-certain in E and hence by our assumption is S-certain in F; thus $F \backslash (P \cap E)$ consists of impossible outcomes and one concludes that $P \cap F \subset E$. One thus deduces the *exchange condition*:

$$P \cap E \subset F \Rightarrow P \cap F \subset E$$

Such subsets P satisfying the exchange condition are called *supports*. For this set of ideas to be coherent, one must admit a degree of ontological uncertainty in the world. Thus, if P is a minimal support and $P \cap E$ is not a singleton, then the choice of one or another outcome in the set $P \cap E$ is to be supposed to be totally without cause and constitute an irreducible ontological indeterminism. We must thus postulate the existence of uncaused acts isolated from the rest of the world, somewhat similarly to the stochastic processes needed to make sense of probabilistic theories. Let us call such isolated indeterminate processes *whimsies*. Imagine a whimsy thus as being a sort of box with n lights and a button. Pushing the bottom makes one of the lights blink, but the choice of which light blinks is to be construed as being ontologically indeterminate and independent of anything else in the world. As for stochastic processes having one such whimsy, by mathematical processing of the results of a run of invocations, one can produce a whimsy with any finite number of outcomes. We shall diagram a whimsy as in Figure 5. Of course one does not necessarily assume the existence of a probability measure associated to the outcomes.

By our thesis, we must now assume that whimsies can become imbedded in experimental procedures and adulterate the pattern of outcomes, giving rise again to the three types of imperfections. To be able to provide concrete examples, we need an analog of an instrumental theory of the probabilistic interpretation. One supposes that one has a certain number of primary

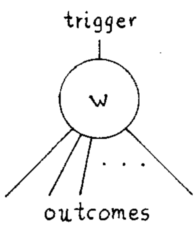

Fig. 5. The whimsy.

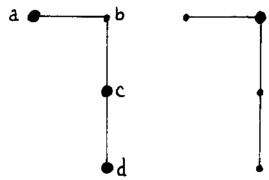

Fig. 6.

ontological states and that at any moment one of these is in fact realized. These primary states thus define a set Σ of supports of any quasimanual $\mathfrak{A}$ that is presumably used to study them. A pair $(\mathfrak{A}, \Sigma)$ of quasimanual and a set of supports whose union is the set of all outcomes is called an *entity* and this is the construct we shall look at.

Consider now the entity given by

$$\mathfrak{A} = \{\{a, b\}, \{b, c, d\}\}$$

$$\Sigma = \{\{a, c, d\}, \{b\}\}$$

which can be conveniently diagrammed as in Figure 6.

A little bit of thought makes us realize that one can view this as follows: there are two ontological states, separated by the single dichotomic operation $\{a, b\}$; the operation $\{b, c, d\}$ can then be interpreted via the construct of Figure 7. Here the difference in the outcomes c and d does not reflect any difference in the ontological state, but merely the functioning of the whimsy. This in fact provides us with an example of a dirty operation.

As the next example, consider the entity

$$\mathfrak{A} = \{\{u, v\}, \{y, z\}, \{a, b, c, d\}\}$$

$$\Sigma = \{\{u, y, a, c\}, \{u, z, a, d\}, \{v, y, b, c\}, \{v, z, b, d\}\}$$

Again, some reflection shows that we can interpret this as dealing with four ontological states completely separated by the two dichotomies $D_1 = \{u, v\}$ and $D_2 = \{y, z\}$ and in which $\{a, b, c, d\}$ has the construct of Figure 8. This

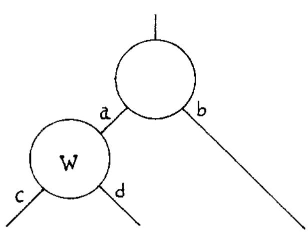

Fig. 7.

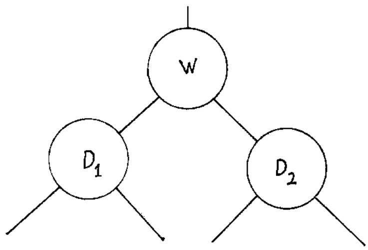

Fig. 8.

is an example of a rotten operation; it cannot be dirty, since a dirty one has to be one or the other of the dichotomies followed by an array of whimsies, and these types cannot distinguish all four ontological states, while this construct can.

As a final example, consider the entity given by the minimal supports of the fly (Figure 9). One can view this in terms of the previous construct by identifying the operations labeled as $\{a, b, c, d\}$, and operations $\{a, b, x\}$ and $\{c, d, x\}$ with two coarsenings. One must motivate, though, the identification of the outcome that coarsens $\{a, b\}$ with the one that coarsens $\{c, d\}$. In the qualitative interpretation, with its minimal realist assumption, outcome identification can be motivated if the respective occurrences furnish equivalent information about the ontological states. In our case, coarsened $\{a, b\}$ and coarsened $\{c, d\}$ both offer *no* information, since they simply signal a whimsy outcome. Thus, identification is justified, and we have succeeded in interpreting the fly entity as a damned theory, since the underlying dichotomies are unavailable to the experimenter, who has only the fly as the quasimanual.

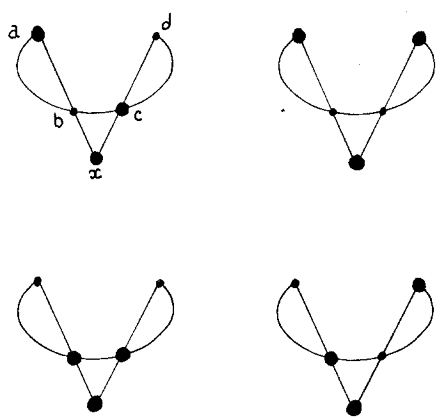

Fig. 9.

9. CONCLUSION

Our aim has been to motivate a mathematical focusing of attention. Existing formalisms provide us with an overabundance of objects. Most of these objects, however, when interpreted as formalized methodology, contain methods interpretable as flawed, and these flaws can be of three types. Mathematical attention should shift to those that are not flawed. In probabilistic quasimanual theory this means identifying those quasimanuals that are of the form $\mathfrak{A}^{\mathrm{ex}}(T)$ for a state-complete canonical theory on some convex set of states. A similar identification of the central problem can be achieved in any other situation once the notions of "dirty," "rotten," and "damned" become explicitly and precisely formulated. Such a program means that one first looks at those "flawless" theories that presuppose that neither Nature nor our scientists are cheating us, that our methods cannot in principle see more than they do, and that the ontological states of affairs in principle are not capable of greater determination than that furnished by our methods. Such theories furnish a framework relative to which all others, incorporating limitations either real (imposed by Nature) or *ad hoc* (imposed by people), can be analyzed.

Perhaps nowhere is the burden of an overabundance of possible objects so apparent as in the great axiomatic effort of Ludwig (1983) to lay the foundations of Hilbert-space quantum mechanics. Roughly 40 axioms are needed to separate quantum mechanics from the general situation. A more satisfying result would be to relate quantum mechanics to its encompassing flawless theory (if it is not already flawless) and to identify axiomatically this latter among all such. The same of course applies for any science.

APPENDIX. MATHEMATICAL DETAILS

Theorem 1. Let W be a finite-dimensional convex polytope; then there is a number n_0 such that any extreme instrument has at most n_0 nonzero elements.

Before giving the proof, we need a few definitions and lemmas which are useful in their own right. Let W^* be the (vector) space of affine functions $W \to \mathbf{R}$. Given $S \subset W^{\mathrm{ex}}$, we say that S is *stiff* if and only if the only affine function that vanishes on S is the zero function. Given $f \in O$, we define

$$Z(f) = W^{\mathrm{ex}} \cap f^{-1}(0)$$

$$U(f) = W^{\mathrm{ex}} \cap f^{-1}(1)$$

$$N(f) = Z(f) \cup U(f)$$

Lemma 1. $f \in O^{\mathrm{ex}} \Leftrightarrow N(f)$ is stiff.

Proof. One has that f is not extreme if and only if it lies on a segment within O, which means that there is an element $h \neq 0$ of W^* such that $f + th \in O$ for $t \in [-1, 1]$. This can only happen if and only if there is a nonzero affine function vanishing on $N(f)$. ■

Lemma 2. $I = (f_1, \ldots, f_n)$ is an extreme instrument if and only if those linear subspaces $V_i = \{h \in W^* | h | N(f_i) = 0\}$ that are not $\{0\}$ are linearly independent.

Proof. I is not extreme if and only if it lies on a segment within O_n, which means that there is a nonzero n-tuple $H = (h_1, \ldots, h_n)$ in W^{*n} such that $\sum_i h_i = 0$ (since elements of instruments sum to 1) and such that $I + tH \in O_n$ for $t \in [-1, 1]$. Clearly, h_i restricted to $N(f_i)$ must be zero. ■

Proof of Theorem 1. Given an extreme instrument $I = (f_1, \ldots, f_n)$, one has from Lemma 1 that $V_i = \{0\}$ if and only if f_i is extreme. Since there is only a finite number of extreme points of O and W^* is finite-dimensional, the result now follows from Lemma 2. ■

Theorem 2. For any instrumental theory T one has $T'' = T'$.

Proof. The instruments of T' are convex combinations of instruments I', where I is extreme in T. Now each such I' is extreme in T', for if it were a convex combination of others, the same convex relation would *a fortiori* hold when restricted to the smaller set $W \subset W'$. Thus, $O_n'^{\text{ex}} = \{I' | I \in O_n^{\text{ex}}\}$. Hence, $\mathfrak{A}^{\text{ex}}(T')$ is isomorphic to $\mathfrak{A}^{\text{ex}}(T)$ and so W'' is isomorphic to W'. ■

Theorem 3. Let $W = S_1 x \cdots x S_k$ be a finite Cartesian product of simplexes; then the canonical theory associated to W is state complete.

We shall first need a few lemmas.

Lemma 3. Let W be an n-simplex; then the canonical theory associated to W is state complete.

Proof. This is essentially proved in Svetlichny (1982, Theorem 3). Each extreme instrument is a coarsening of the single maximal extreme instrument given by the set of barycentric coordinates of W. One immediately has that $W' = W$. ■

Lemma 4. Let C and D be finite-dimensional, compact, convex sets. Any frequency functional on $C \times D$ is of the form $f + g$, where f is a frequency functional on C and g is a frequency functional on D.

Proof. Let h be any frequency functional on $C \times D$, and (c, d), (c', d') any two points. Then (c, d') and (c', d) are also points of $C \times D$ and one has

$$\tfrac{1}{2}(c, d) + \tfrac{1}{2}(c', d') = \tfrac{1}{2}(c', d) + \tfrac{1}{2}(c, d')$$

By affinity of h one now deduces

$$h(c, d) + h(c', d') = h(c', d) + h(c, d')$$

By compactness of $C \times D$ there is a point (c', d') at which h takes its minimum; thus, defining $f(c) = h(c, d')$ and $g(d) = h(c', d) - h(c', d')$, one has the desired representation. ■

Lemma 5 Let C and D be finite-dimensional, compact, convex sets. The extreme instrument quasimanual of the canonical theory associated to $C \times D$ is the disjoint union of the extreme instrument quasimanuals of the canonical theories associated to each one of the factors.

Proof. Let $(h_1, \ldots, h_n) = (f_1 + g_1, \ldots, f_n + g_n)$ be an extreme instrument of the theory associated to $C \times D$. One has that for all (c, d), $\sum_i f_i(c) = 1 - \sum_i g_i(d)$. Thus, one has that $\sum_i f_i(c) = u$ and $\sum_i g_i(d) = v$ for some positive u and v with $u + v = 1$. Suppose now that $0 < u, v < 1$. Then there is a positive number p such that each of the $(1+p)f_i$, $(1 - pu/v)g_i$, $(1-p)f_i$, $(1 + pu/v)g_i$,

$$h_i^+ = (1+p)f_i + (1 - pu/v)g_i$$

$$h_i^- = (1-p)f_i + (1 + pu/v)g_i$$

is a frequency functional. One now has that $h = h^+/2 + h^-/2$ and thus is not extreme. Thus either each f_i or each g_i is zero. ■

Proof of Theorem 3. The case of a single simplex follows from Lemma 3; the general case now follows by induction on the number of factors from Lemma 5. ■

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